

COMPACT SPACES AS QUOTIENTS OF PROJECTIVE FRAÏSSÉ LIMITS

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ABSTRACT. We develop a theory of projective Fraïssé limits in the spirit of Irwin-Solecki, which moreover support a dual structure. Let K be a zero-dimensional, second countable, compact space. We prove that a subgroup G of $\text{Homeo}(K)$ is closed in the compact-open topology if and only if it is the automorphism group of some dual topological Fraïssé limit \mathbf{K} on domain K . As an application we prove that every second countable, compact space is the quotient of a projective Fraïssé limit \mathbf{K} with a closed equivalence relation on K that is definable in \mathbf{K} .

INTRODUCTION

The interplay between Fraïssé theory and dynamics of non-Archimedean Polish groups established by the correspondence between closed subgroups of S_∞ and countable Fraïssé structures has been fruitful in many ways (see for example the survey [Kec12]). Taking this fact as a departure point it is natural to seek what would the natural correspondence be for projective Fraïssé structures.

Projective Fraïssé structures were first introduced by T. Irwin and S. Solecki in [ISS06]. These are zero-dimensional, compact, topological structures which provide a dual to the classical Fraïssé structures. However, the duality in [ISS06] was not explicitly present in the structure as such but rather in the morphisms that were used in the amalgamating process. In the same paper they used the projective limit of finite, reflexive, linear graphs in language $\mathcal{L}_R = \{R\}$, where R is a binary relation symbol, to provide a simple description of the unique hereditary, indecomposable, chainable continuum known as the pseudo-arc. This enabled them to deduce new results regarding the pseudo-arc and to show how previously known facts about it (e.g., projective universality and homogeneity) were linked at a deeper level.

Since then, projective Fraïssé constructions were used by several authors in the study of compact spaces and their homeomorphisms groups e.g., [BK13], [Kwi12], [Kwi13]. There is standard scheme in the analysis that appears in these papers. One starts with a second-countable compact space Y where either Y or the group $\text{Homeo}(Y)$ is under investigation. Then one defines an appropriate class \mathcal{K} of finite model theoretic \mathcal{L} -structures which “approximate” Y . If this class \mathcal{K} satisfies the “projective Fraïssé” axioms [ISS06] then the projective Fraïssé limit \mathbf{M} of \mathcal{K} is uniquely defined. Since the domain M of \mathbf{M} is a zero-dimensional space one cannot expect to equate it with Y which is in general not zero-dimensional. However, in all the cases cited above the structures appearing in \mathcal{K} were classes of graphs perhaps

with extra model theoretic structure added on top. The edge relation R was part of the language \mathcal{L} and in the limit \mathbf{M} the relation $R^{\mathbf{M}}$ always turned out to be an equivalence relation. The quotient $\mathbf{M}/R^{\mathbf{M}}$ was equal to Y and the map $\text{Aut}(\mathbf{M}) \rightarrow \text{Homeo}(Y)$ that was induced by the quotient was a continuous embedding with dense image in $\text{Homeo}(Y)$. This allowed the study of Y and $\text{Homeo}(Y)$ via the combinatorial properties of the Fraïssé class \mathcal{K} .

The purpose of this note is to turn this observations to a general method. However, this is not always possible if we insist to use only classical model theoretic structure as done in [ISS06]. We rewrite the theory developed in [ISS06] dualizing it not only in the level of the morphisms but also in the level of structures i.e. we also allow dual predicates in \mathcal{L} which quantify over dual tuples (finite clopen partitions). We will call a topological \mathcal{L} -structure purely dual if \mathcal{L} contains only dual predicates.

In Chapter 3 we use a standard orbit completion argument (i.e., we add extra predicates; one for every orbit) and equipping $\text{Homeo}(K)$ with the compact-open topology we get the following result.

Theorem 3.1. *Let G be a closed subgroup of $\text{Homeo}(K)$ where K is zero-dimensional, compact, second-countable space. Then there is a purely dual projective Fraïssé structure \mathbf{K} on domain K such that $\text{Aut}(\mathbf{K}) = G$.*

Theorem 3.1 should be thought of as the dual of the statement that wants every closed subgroup of S_∞ to be the automorphism group of a classical Fraïssé limit. In Chapter 4 we provide a method which allows to turn any topological \mathcal{L} -structure into a purely dual one without losing any information. We also show that above theorem is false if we do not allow our structures to support a dual structure. Therefore the context of dual structures is strictly more general then the context of direct structures.

In Chapter 5 we fix a special binary relation symbol \mathfrak{r} whose interpretation will always be a reflexive and symmetric closed relation. This should paralleled with the metric Fraïssé theory where the symbol d is reserved as a signifier for the metric. A formal relational language \mathcal{L} will be decorated with the subscript \mathfrak{r} whenever \mathfrak{r} belongs to \mathcal{L} . Therefore, we always have $\mathfrak{r} \in \mathcal{L}_{\mathfrak{r}}$ and $\mathcal{L}_{\mathfrak{r}}$ -structures are always in particular reflexive \mathfrak{r} -graphs. We say that an $\mathcal{L}_{\mathfrak{r}}$ -structure \mathbf{K} is a pre-space if $\mathfrak{r}^{\mathbf{K}}$ is moreover transitive and therefore an equivalence relation. We apply Theorem 3.1 to get the following result.

Theorem 5.2. *For every second-countable, compact space Y there is a projective Fraïssé pre-space \mathbf{K} such that:*

$$(K/\mathfrak{r}^{\mathbf{K}}) \cong_{\text{homeo}} Y.$$

Moreover, the map $\text{Aut}(\mathbf{K}) \rightarrow \text{Homeo}(Y)$ induced by the quotient map $K \rightarrow (K/\mathfrak{r}^{\mathbf{K}})$ is a continuous group embedding with dense image in $\text{Homeo}(Y)$.

We shall note here that R. Camerlo characterized all possible quotients M/\mathfrak{r}^M of Fraïssé structures in the language $\{\mathfrak{r}\}$ to be certain combinations of singletons, Cantor spaces and pseudo-arcs [RC10].

Acknowledgements. This set of notes includes –I think– the first written account of Fraïssé theory with dual predicates. This is not however the original idea of this paper. The idea of using dual predicates was first introduced in a systematic way by S. Solecki [Sł10, Sł12] in Ramsey theory to formalize dual Ramsey statements such as the Graham and Rothschild theorem [GR71]. S. Solecki had known for a fact that projective Fraïssé theory can be developed including dual predicates and he was the one who pointed out that dual structure might be needed for the proof of Theorem 3.1. An example in Chapter 4 shows that for this theorem to be true dual predicates are actually necessary. I would like to thank Sławek since many of the ideas included here grew out of the discussions we had. I would also like to thank Ola Kwiatkowska for sharing with me her insight on this problem and Gianluca Basso for his useful comments.

1. PRELIMINARIES

In what follows, K will always denote a zero-dimensional, compact, second-countable space. Our main objects of study will be spaces K as above, which support both the usual model-theoretic structure as well as dual structure. To make this precise, following S. Solecki [Sł10, Sł12], we consider the following two types of tuples.

- In the classical model-theoretic context, a tuple of size $n > 0$ in K corresponds to an injection

$$i : \{0, \dots, n-1\} \rightarrow K$$

We will call this kind of tuple a *direct tuple*. We denote the set of all direct tuples in K of size $n > 0$ by K^n .

- In the dual context, a tuple of size n in K corresponds to a surjection

$$e : K \rightarrow \{0, \dots, n-1\}$$

Since our intention is to work with “topological” structures, we endow $\{0, \dots, n-1\}$ with the discrete topology and we impose a further regularity condition, that e is also continuous. We will call this kind of tuple a *dual tuple*. We denote the set of all dual tuples in K of size n by n^K .

Notice that for K zero-dimensional, compact, second countable space, the set n^K is at most countably infinite and moreover these functions suffice to separate points of K , i.e., for every $x_1, x_2 \in K$ there is an $e \in n^K$ such that $e(x_1) \neq e(x_2)$. Notice also that the set n^K of all dual tuples naturally corresponds to the set $\text{CP}_n(K)$ of all clopen, ordered, n -partitions of K , i.e.,

$$\text{CP}_n(K) = \{(\Delta_0, \dots, \Delta_{n-1}) : \Delta_i \subset K \text{ clopen, } \Delta_i \cap \Delta_j = \emptyset \text{ } \cup_i \Delta_i = K\}.$$

Whenever it is convenient for notational purposes we will not distinguish between the set $\{\Delta_0, \dots, \Delta_{n-1}\}$ and the tuple $(\Delta_0, \dots, \Delta_{n-1})$. If for example $P \in \text{CP}_n(K)$ and Δ is a clopen set appearing some of the n entries of P then we will write $\Delta \in P$.

1.1. Topological \mathcal{L} -structures. We will work with relational only languages \mathcal{L} . To each relational symbol R in \mathcal{L} corresponds some natural number $\text{arity}(R) > 0$ which is the arity of the symbol R . Moreover, for every symbol in \mathcal{L} we have predetermined our intention to use it in the direct or in the dual context. We will make the convention here of using lower case letters r, p, q, \dots for direct relational symbols, and capital letters R, P, Q, \dots for dual relational symbols. We will call the language \mathcal{L} *purely direct* if it contains only direct symbols and *purely dual* if it contains only dual symbols.

By a *topological \mathcal{L} -structure \mathbf{K}* we mean a zero-dimensional, second countable, compact space K together with appropriate interpretations for every symbol in \mathcal{L} .

- If $r \in \mathcal{L}$ is a direct relation symbol of arity n , an appropriate interpretation for r is any closed subset $r^{\mathbf{K}}$ of K^n .
- If $R \in \mathcal{L}$ is a dual relation symbol of arity n , an appropriate interpretation for R is any subset $R^{\mathbf{K}}$ of n^K , or equivalently, any subset of $\text{CP}_n(K)$.

We call a topological \mathcal{L} -structure *purely direct \mathcal{L} -structure* whenever \mathcal{L} is purely direct and *purely dual \mathcal{L} -structure* whenever \mathcal{L} is purely dual.

1.2. Morphisms. In the classical model theoretic context the central morphism is the embedding. Here however we will be interested mostly in dual morphisms, the epimorphisms. Let \mathbf{A}, \mathbf{B} be two dual topological \mathcal{L} -structure. By an *epimorphism* f from \mathbf{A} to \mathbf{B} we mean a continuous surjection $f : A \rightarrow B$ such that:

- for every $r \in \mathcal{L}$ of arity say m and every $\beta \in B^m$ we have

$$\beta \in r^{\mathbf{B}} \iff \exists \alpha \in r^{\mathbf{A}} \quad \beta = f \circ \alpha$$

- for every $R \in \mathcal{L}$ of arity say m and for every $\beta \in m^B$ we have

$$\beta \in R^{\mathbf{B}} \iff \beta \circ f \in R^{\mathbf{A}}$$

An isomorphism between \mathbf{A} and \mathbf{B} is a bijective epimorphism and an automorphism of \mathbf{A} is an isomorphism from \mathbf{A} to \mathbf{A} .

1.3. Induced structures and factoring through maps. Let \mathbf{K} be a topological \mathcal{L} -structure, let A be a zero-dimensional, second countable, compact space and let $f : K \rightarrow A$ be a continuous surjection. Notice that there is a unique topological \mathcal{L} -structure \mathbf{A} on domain A that renders f an epimorphism. We call this structure \mathbf{A} , the structure *induced by the map f* .

Let $f : \mathbf{K} \rightarrow \mathbf{A}, h : \mathbf{K} \rightarrow \mathbf{B}$ be epimorphisms. We say that f *factors through* h if there is an epimorphism $f_h : \mathbf{B} \rightarrow \mathbf{A}$ such that $f_h \circ h = f$.

Lemma 1.1. *Let \mathbf{A}, \mathbf{B} and \mathbf{K} be topological \mathcal{L} -structures with \mathbf{A}, \mathbf{B} finite. Let also $f : \mathbf{K} \rightarrow \mathbf{A}$ and $g : \mathbf{K} \rightarrow \mathbf{B}$ be epimorphisms. Then there is a finite topological \mathcal{L} -structure \mathbf{C} and an epimorphism $h : \mathbf{K} \rightarrow \mathbf{C}$ such that both f and g factor through h .*

Proof. Let $C = (\Delta_0, \dots, \Delta_{n-1})$ a clopen partition of K whose every entry Δ_i is a (nonempty) set of the form $f^{-1}(a) \cap g^{-1}(b)$, where $a \in A$ and $b \in B$. Let $h : K \rightarrow C$ be the inclusion map, i.e., $h(x) = \Delta_i$ if and only if $x \in \Delta_i$. This map is a continuous surjection, so, it induces a structure \mathbf{C} on domain C . It is immediate now that both f and g factor through h . \square

1.4. The inverse limit construction. Given a sequence $\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_i, \dots$ of finite topological \mathcal{L} -structures together with epimorphisms $\pi_i : \mathbf{A}_{i+1} \rightarrow \mathbf{A}_i$, we can define a new structure \mathbf{M} and epimorphisms $\pi_i^\infty : \mathbf{M} \rightarrow \mathbf{A}_i$ through an inverse limit construction. Let

$$M = \{(a_1, a_2, \dots) \in \prod_{i \in \mathbb{N}} A_i : \forall i \geq 1 \pi_i(a_{i+1}) = a_i\}.$$

M is a closed subset of the compact space $\prod_{i \in \mathbb{N}} A_i$ and it will serve as the domain of \mathbf{M} . We define π_i^∞ to be the projection map from M to A_i .

For $r \in \mathcal{L}$ of arity say m , and $\beta \in M^m$ we let $\beta \in r^{\mathbf{M}}$ if and only if $\pi_i^\infty \circ \beta \in r^{\mathbf{A}_i}$ for all $i \in \mathbb{N}$. For $R \in \mathcal{L}$ of arity say m , and $\gamma \in m^M$, notice that there is an $i_0 \in \mathbb{N}$ such that γ factors through $\pi_{i_0}^\infty$. Let $\alpha \in m^{\mathbf{A}_{i_0}}$ be such that $\gamma = \alpha \circ \pi_{i_0}^\infty$. We let $\gamma \in R^{\mathbf{M}}$ if and only if $\alpha \in R^{\mathbf{A}_{i_0}}$, which happens if and only if for every $i > i_0$ we have $(\alpha \circ \pi_{i_0} \circ \dots \circ \pi_{i-1}) \in R^{\mathbf{A}_i}$.

This turns \mathbf{M} into a topological \mathcal{L} -structure and every π_i^∞ to an epimorphism. We call \mathbf{M} the *inverse limit* of the *inverse system* $\{(\mathbf{A}_i, \pi_i) : i \in \mathbb{N}\}$ and we write $\mathbf{M} = \varprojlim (\mathbf{A}_i, \pi_i)$.

2. PROJECTIVELY FRAÏSSÉ STRUCTURES

In Chapter 7 of [Hod93], Hodges reviews the theory of Fraïssé limits of direct structures via direct morphisms (embeddings). Following Hodges and [ISS06] we present here the theory of Fraïssé limits of topological \mathcal{L} -structures via dual morphisms. To avoid confusion, we should emphasize two things. First, what in [ISS06] is called topological \mathcal{L} -structure, here it falls under the name purely direct topological \mathcal{L} -structure. Second, in [ISS06] a projective Fraïssé class, in contrast with the definition that we will be using here, is not bound to satisfy the hereditary property (HP).

We say that a topological \mathcal{L} -structure \mathbf{M} is *projectively Fraïssé* or *projectively ultra-homogeneous* if for every two epimorphisms f_1, f_2 of \mathbf{M} on some finite topological \mathcal{L} -structure \mathbf{A} there is an automorphism g of \mathbf{M} such that $f_1 \circ g = f_2$.

For every topological \mathcal{L} -structure \mathbf{M} we denote by $\text{Age}(\mathbf{M})$ the class of all the finite topological \mathcal{L} -structures \mathbf{A} such that \mathbf{M} epimorphs on \mathbf{A} . We call a class \mathcal{K} of topological \mathcal{L} -structures an *age* if $\mathcal{K} = \text{Age}(\mathbf{M})$ for some topological \mathcal{L} -structure

M. It is immediate that if \mathcal{K} is an age, then \mathcal{K} is not empty, any subclass of \mathcal{K} of pairwise non-isomorphic structures is at most countable, and the following two properties hold for \mathcal{K} .

- *Hereditary Property* (HP): if $\mathbf{A} \in \mathcal{K}$ and \mathbf{A} epimorphs onto a structure \mathbf{B} , then $\mathbf{B} \in \mathcal{K}$.
- *Joint Surjecting Property* (JSP): if $\mathbf{A}, \mathbf{B} \in \mathcal{K}$ then there is $\mathbf{C} \in \mathcal{K}$ that epimorphs onto both \mathbf{A} and \mathbf{B} .

The converse is also true i.e. if \mathcal{K} is a non empty class of finite topological \mathcal{L} -structures such that any subclass of \mathcal{K} of pairwise non-isomorphic structures is at most countable and the above two properties hold for \mathcal{K} then \mathcal{K} is an age.

To see this, let $\mathbf{A}_1, \mathbf{A}_2, \dots$ be a list of structures in \mathcal{K} that up to isomorphism exhaust \mathcal{K} . Using the JSP we can find a new list $\mathbf{B}_1, \mathbf{B}_2, \dots$ of structures in \mathcal{K} such that $\mathbf{B}_1 = \mathbf{A}_1$ and for $i > 1$, \mathbf{B}_{i+1} epimorphs on both \mathbf{A}_{i+1} and \mathbf{B}_i . Let $\pi_i : \mathbf{B}_{i+1} \rightarrow \mathbf{B}_i$ be such epimorphisms and let $\mathbf{M} = \varprojlim (\mathbf{B}_i, \pi_i)$. Then $\mathcal{K} = \text{Age}(\mathbf{M})$ because by construction \mathbf{M} epimorphs to every \mathbf{A}_i and moreover, every epimorphism of \mathbf{M} to some finite dual topological \mathcal{L} -structure \mathbf{A} factors through an epimorphism from some $\mathbf{B}_i \in \mathcal{K}$ that was used in the inverse system so, by HP we have that $\mathbf{A} \in \mathcal{K}$.

Given now that the structure \mathbf{M} is projectively ultr-ahomogeneous, it is easy to see that its age \mathcal{K} satisfies moreover the following property.

- *Projective Amalgamation Property* (PAP): if $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{K}$ and $f_A : \mathbf{A} \rightarrow \mathbf{C}$, $f_B : \mathbf{B} \rightarrow \mathbf{C}$ are epimorphisms, then there is $\mathbf{D} \in \mathcal{K}$ and epimorphisms $g_A : \mathbf{D} \rightarrow \mathbf{A}$, $g_B : \mathbf{D} \rightarrow \mathbf{B}$ such that $f_A \circ g_A = f_B \circ g_B$.

To check that this is true, notice first that since $\mathbf{A}, \mathbf{B} \in \mathcal{K}$, there are epimorphisms $h_A : \mathbf{M} \rightarrow \mathbf{A}$ and $h_B : \mathbf{M} \rightarrow \mathbf{B}$. But then $f_A \circ h_A$ and $f_B \circ h_B$ are both epimorphisms from \mathbf{M} to \mathbf{C} . So, by projective ultra-homogeneity of \mathbf{M} there is $\phi \in \text{Aut}(\mathbf{M})$ such that $f_A \circ h_A \circ \phi = f_B \circ h_B$. Using Lemma 1.1, we can find $\mathbf{D} \in \mathcal{K}$ and an epimorphism $h_D : \mathbf{M} \rightarrow \mathbf{D}$ such that h_A and $h_B \circ \phi^{-1}$ factor through h_D . Let $g_A : \mathbf{D} \rightarrow \mathbf{A}$ and $g_B : \mathbf{D} \rightarrow \mathbf{B}$ be the maps that close these diagrams, i.e., $g_A \circ h_D = h_A$ and $g_B \circ h_D = h_B \circ \phi^{-1}$. The functions g_A and g_B are the required epimorphisms from \mathbf{D} to \mathbf{A} and \mathbf{B} in respect.

In Theorem 2.3 we will see that the converse is also true, i.e., if an age \mathcal{K} has PAP then we can built from it a projective Fraïssé structure \mathbf{M} with $\text{Age}(\mathbf{M}) = \mathcal{K}$. An age \mathcal{K} that satisfies PAP is called *projective Fraïssé class*.

Let \mathbf{M} be a topological \mathcal{L} -structure with $\text{Age}(\mathbf{M}) = \mathcal{K}$. We say that \mathbf{M} has the *finite extension property* if for every $\mathbf{A}, \mathbf{B} \in \mathcal{K}$ and $f : \mathbf{B} \rightarrow \mathbf{A}$, $g : \mathbf{M} \rightarrow \mathbf{A}$ epimorphisms, there is an epimorphism $h : \mathbf{M} \rightarrow \mathbf{B}$ such that $f \circ h = g$. We say that \mathbf{M} has the *one point extension property* if the above holds when the size of \mathbf{B} is one more than the size of \mathbf{A} . Notice that for any topological \mathcal{L} -structure \mathbf{M} , \mathbf{M} has the one point extension property if and only if \mathbf{M} has the finite extension property.

Lemma 2.1. *Let \mathbf{M} and \mathbf{N} be two topological \mathcal{L} -structure of the same age \mathcal{K} . Let $\mathbf{A} \in \mathcal{K}$ and let $f : \mathbf{M} \rightarrow \mathbf{A}$ and $g : \mathbf{N} \rightarrow \mathbf{A}$ be two epimorphisms. If \mathbf{M} and \mathbf{N} have the finite extension property then there is an isomorphism $h : \mathbf{M} \rightarrow \mathbf{N}$ such that $g \circ h = f$.*

Proof. We will use a back and forth type of argument. For every $n \in \mathbb{N}$ we will construct $\mathbf{A}_n \in \mathcal{K}$ and epimorphisms $f_n : \mathbf{M} \rightarrow \mathbf{A}_n$, $g_n : \mathbf{N} \rightarrow \mathbf{A}_n$, and for every $n > 0$ we will also construct an epimorphism $\pi_{n-1} : \mathbf{A}_n \rightarrow \mathbf{A}_{n-1}$. At the end of the construction \mathbf{M} and \mathbf{N} will be proven to be isomorphic to $\varprojlim(\mathbf{A}_n, \pi_n)$. By using these indirect isomorphisms we will get the desired isomorphism h . Let $\{e_n : n \in \mathbb{N}\}$ be an enumeration of dual tuples m^M of M for every $m > 0$ and let $\{e'_n : n \in \mathbb{N}\}$ be an enumeration of dual tuples m^N of N for every $m > 0$.

$n = 0$. Let $\mathbf{A}_0 = \mathbf{A}$, $f_0 = f$ and $g_0 = g$.

odd $n > 0$. Using Lemma 1.1 we can find a structure \mathbf{A}_n and an epimorphism $f_n : \mathbf{M} \rightarrow \mathbf{A}_n$ such that both f_{n-1} and e_{n-1} factor through f_n . Let $\pi_{n-1} : \mathbf{A}_n \rightarrow \mathbf{A}_{n-1}$ be the epimorphism that closes the one diagram, i.e., $\pi_{n-1} \circ f_n = f_{n-1}$. Finally define $g_n : \mathbf{N} \rightarrow \mathbf{A}_n$ to be any map such that $\pi_{n-1} \circ g_n = g_{n-1}$. A map like this exists, since \mathbf{N} satisfies the finite extension property.

even $n > 0$. Again, using Lemma 1.1 we can find a structure \mathbf{A}_n and an epimorphism $g_n : \mathbf{N} \rightarrow \mathbf{A}_n$ such that both g_{n-1} and e'_{n-1} factor through g_n . Let $\pi_{n-1} : \mathbf{A}_n \rightarrow \mathbf{A}_{n-1}$ be the epimorphism that closes the one diagram, i.e., $\pi_{n-1} \circ g_n = g_{n-1}$. Finally define $f_n : \mathbf{M} \rightarrow \mathbf{A}_n$ to be any map such that $\pi_{n-1} \circ f_n = f_{n-1}$. A map like this exists, since \mathbf{M} satisfies the finite extension property.

Let now $\mathbf{B} = \varprojlim(\mathbf{A}_n, \pi_n)$. The maps $\mu : \mathbf{M} \rightarrow \mathbf{B}$ with $\mu(x) = (f_0(x), f_1(x), \dots)$ and $\nu : \mathbf{N} \rightarrow \mathbf{B}$ with $\nu(x) = (g_0(x), g_1(x), \dots)$ are bijections since the families $\{e_n\}$ and $\{e'_n\}$ separate points of M and N in respect. It is moreover easy to see that μ and ν are actually isomorphisms. So, the map $h : \mathbf{M} \rightarrow \mathbf{N}$ with $h = \nu^{-1} \circ \mu$ is also an isomorphism which by construction satisfies the desired property $g \circ h = f$. \square

Lemma 2.2. *Let \mathbf{M} be a topological \mathcal{L} -structure with $\text{Age}(\mathbf{M}) = \mathcal{K}$ then the following are equivalent:*

- (1) \mathbf{M} is projectively ultra-homogeneous;
- (2) \mathbf{M} has the finite extension property;
- (3) \mathbf{M} has the one point extension property.

Proof. It is immediate that (2) and (3) are equivalent. We prove that (1) is also equivalent to (2).

(1) \rightarrow (2) Let $\mathbf{A}, \mathbf{B} \in \mathcal{K}$ and $f : \mathbf{B} \rightarrow \mathbf{A}$, $g : \mathbf{M} \rightarrow \mathbf{A}$ epimorphisms. Since $\mathbf{B} \in \mathcal{K} = \text{Age}(\mathbf{M})$, there is an epimorphism $j : \mathbf{M} \rightarrow \mathbf{B}$. So, $f \circ j : \mathbf{M} \rightarrow \mathbf{A}$ is an epimorphism, and by the projective ultra-homogeneity of \mathbf{M} there is $\phi \in \text{Aut}(\mathbf{M})$ with $g \circ \phi = f \circ j$. Let $h = j \circ \phi^{-1}$. Then $h : \mathbf{M} \rightarrow \mathbf{B}$ is an epimorphism such that $f \circ h = g$.

(2) \rightarrow (1) Let $f_1, f_2 : \mathbf{M} \rightarrow \mathbf{A}$ be epimorphisms for some $\mathbf{A} \in \mathcal{K}$. Then by Lemma 2.1, there is $g \in \text{Aut}(\mathbf{M})$ such that $f_1 \circ g = f_2$. \square

Theorem 2.3. *For every projective Fraïssé class \mathcal{K} there is a unique, up to isomorphism, projectively ultra-homogeneous topological \mathcal{L} -structure \mathbf{M} such that $\text{Age}(\mathbf{M}) = \mathcal{K}$.*

Proof. First notice that $\mathbf{M}_1, \mathbf{M}_2$ share the same age and are both projectively ultra-homogeneous, by Lemma 2.2 they have finite extension property. Let \mathbf{A} any structure in \mathcal{K} . Since \mathcal{K} is the age of both $\mathbf{M}_1, \mathbf{M}_2$, there are epimorphisms $f_1 : \mathbf{M}_1 \rightarrow \mathbf{A}$ and $f_2 : \mathbf{M}_2 \rightarrow \mathbf{A}$. Lemma 2.1 gives as then an isomorphism h between \mathbf{M}_1 and \mathbf{M}_2 . \square

3. CLOSED SUBGROUPS OF $\text{Homeo}(K)$

By definition and since K is compact, every automorphism of a topological \mathcal{L} -structure \mathbf{K} is also a homeomorphism, therefore, $\text{Aut}(\mathbf{K})$ can be seen as a subgroup of $\text{Homeo}(K)$. We will view $\text{Homeo}(K)$ as a topological group equipped with the compact-open topology τ_{co} . The collection of the sets

$$V(F, U) = \{g \in \text{Homeo}(K) : g(F) \subset U\},$$

where F is a compact subset of K and U is an open subset of K , provide a subbase for τ_{co} . In this topology the group $\text{Aut}(\mathbf{K})$ of automorphisms of a dual topological \mathcal{L} -structure \mathbf{K} is a closed subgroup of $\text{Homeo}(K)$. To check this, let $g \notin \text{Aut}(\mathbf{K})$. We will find an open neighborhood V_g of g in $\text{Homeo}(K)$ which does not intersect $\text{Aut}(\mathbf{K})$. Since $g \notin \text{Aut}(\mathbf{K})$, one of the following holds:

- (1) there is $R \in \mathcal{L}$ of arity say m and a dual tuple $e \in m^K$ such that $\mathbf{K} \models R(e)$ if and only if $\mathbf{K} \not\models R(e \circ g^{-1})$, or
- (2) there is $r \in \mathcal{L}$ of arity say m and a tuple $i \in K^m$ such that $\mathbf{K} \models r(i)$ but $\mathbf{M} \not\models r(g \circ i)$, or
- (3) there is $r \in \mathcal{L}$ of arity say m and a tuple $i \in K^m$ such that $\mathbf{K} \not\models r(i)$ but $\mathbf{M} \models r(g \circ i)$.

In the first case notice that if $g \notin \text{Aut}(\mathbf{K})$ then there is $R \in \mathcal{L}$ of arity say m and But then $V(e^{-1}(0), g \circ e^{-1}(0)) \cap \dots \cap V(e^{-1}(m-1), g \circ e^{-1}(m-1))$ is an open subset of $\text{Homeo}(K)$ containing g and lying entirely out of $\text{Aut}(\mathbf{K})$.

In the second case, because $r^{\mathbf{M}}$ is closed, we can find an open rectangle $U_0 \times \dots \times U_{m-1}$ around $((g \circ i)(0), \dots, (g \circ i)(m-1))$ which does not intersect $r^{\mathbf{M}}$. Therefore, let $V_g = V(\{i(0)\}, U_0) \cap \dots \cap V(\{i(m-1)\}, U_{m-1})$.

For the last case, notice that if we let $(b_0, \dots, b_{m-1}) = ((g \circ i)(0), \dots, (g \circ i)(m-1))$, then, as in the previous case we can find open neighborhood $V_{g^{-1}}$ of g^{-1} such that for every $f \in V_{g^{-1}}$, $f(b_0, \dots, b_{m-1}) \notin r^{\mathbf{M}}$. Let then $V_g = V_{g^{-1}}^{-1} = \{f^{-1} : f \in V_{g^{-1}}\}$. Using the continuity of the inversion operator $f \rightarrow f^{-1}$ in τ_{co} we have that V_g is open and moreover $g \in V_g \subset \text{Aut}(\mathbf{K})^c$.

The following theorem says that the inverse of the above observation is true, i.e., for every closed subgroup G of $\text{Homeo}(K)$ there is topological \mathcal{L} -structure \mathbf{K}

on K such that $G = \text{Aut}(\mathbf{K})$. Moreover, \mathbf{K} can be taken to be purely dual and projectively ultra-homogeneous.

Theorem 3.1. *Let G be a closed subgroup of $\text{Homeo}(K)$. Then there is a purely dual projective Fraïssé structure \mathbf{K} on domain K such that $\text{Aut}(\mathbf{K}) = G$.*

Proof. For every $n > 0$, the group G acts on n^K in a natural way: for $g \in G$ and $e \in n^K$ let

$$g \cdot e := e \circ g^{-1}.$$

We denote this action by $G \curvearrowright n^K$. Notice that this action corresponds to the following action $G \curvearrowright \text{CP}_n$ of G on CP_n : for $g \in G$ and $P = (\Delta_0, \dots, \Delta_{n-1}) \in \text{CP}_n(K)$ let

$$g \cdot P := (g(\Delta_0), \dots, g(\Delta_{n-1})).$$

For each $n > 0$ let $(\mathcal{O}_i^n : i \in I_n)$ be the collection of all orbits of $G \curvearrowright n^K$.

Consider now the language $\mathcal{L} = \bigcup_{i=1}^{\infty} \mathcal{L}^n$, where \mathcal{L}^n is the language that consists of n -ary relational symbols $\{O_i^n : i \in I_n\}$, one for every orbit \mathcal{O}_i^n . We turn K into a topological \mathcal{L} -structure \mathbf{K} . For $e \in m^K$ we let

$$\mathbf{K} \models O_i^m(e) \quad \text{if and only if} \quad e \in \mathcal{O}_i^m.$$

It is immediate that $G \subseteq \text{Aut}(\mathbf{K})$. We work now towards the converse inclusion.

Let $g \in \text{Aut}(\mathbf{K})$ and let $V(F, U)$ be an open neighborhood of g in $\text{Homeo}(K)$. We can assume that $U \neq K$. We will find $h \in G \cap V(F, U)$ which will prove that $G \supseteq \text{Aut}(\mathbf{K})$. Because $g(F)$ is compact and U is a union of clopen sets, $g(F)$ can be covered with finitely many of them, so we can assume without loss of generality that U is clopen and $U \neq K$. Notice that $g \in V(g^{-1}(U), U) \subset V(F, U)$. Consider the following two dual tuples $e_1, e_2 \in 2^K$, with $e_1^{-1}(\{0\}) = g^{-1}(U)$, $e_1^{-1}(\{1\}) = K \setminus g^{-1}(U)$ and $e_2^{-1}(\{0\}) = U$, $e_2^{-1}(\{1\}) = K \setminus U$. Since g is an automorphism of \mathbf{K} and since $e_1 = g \cdot e_2$, we have that e_1 and e_2 lie in the same orbit \mathcal{O}_i^2 for some $i \in I_2$. Therefore, there is an $h \in G$ that sends $g^{-1}(U)$ into U and therefore $h \in G \cap V(F, U)$, which proves that $G = \text{Aut}(\mathbf{K})$.

We prove now that \mathbf{K} is projectively Fraïssé. First notice that for every dual tuple $e \in m^K$, there is a unique $i \in I_m$ such that $\mathbf{K} \models O_i^m(e)$. Let $\mathbf{C} \in \mathbf{K}$ and let f_1, f_2 be two epimorphisms of \mathbf{K} onto \mathbf{C} . We can assume without the loss of generality that $C = \{0, \dots, m-1\}$ for some $m > 0$ and therefore $f_1, f_2 \in m^K$. Because f_1 and f_2 induce the same structure \mathbf{C} , there is a unique $i \in I_m$ such that $\mathbf{K} \models O_i^m(f_1)$ and $\mathbf{K} \models O_i^m(f_2)$. Therefore, f_1 and f_2 lie in the same orbit $G \curvearrowright m^K$, so there is $g \in \text{Aut}(\mathbf{K})$ such that $f_1 \circ g = f_2$, showing that \mathbf{K} is projectively ultra-homogeneous. \square

4. TURNING A STRUCTURE TO A PURELY DUAL ONE

We defined our structures so that they carry both dual and direct structure. Here we show that it is always possible to translate the direct structure into a dual one without losing any information. We provide a counterexample to show that the

converse is not always possible. From the beginning, therefore, we could have had developed all of our theory for purely dual topological \mathcal{L} -structures which seem to be the appropriate structures to consider when working with dual morphisms. However, in Chapter 5 it will be convenient to make use of direct relations even if the same things could be done with dual relations only. Moreover, there are many examples of structures whose most natural presentation would involve both direct and dual structure.

Let \mathcal{L} be a language and \mathbf{M} a topological \mathcal{L} -structure. Let also $s \in \mathcal{L}$ be a direct relation of arity n . For every k with $0 < k \leq n$ and for every $f \in k^n$ (f is therefore a surjection), we introduce a dual relational symbol R_s^f of arity $k + 1$. Let

$$\mathcal{L}_s = \mathcal{L} \cup \{R_s^f : f \in k^n \text{ for some } 0 < k \leq n\} \setminus \{s\}.$$

We turn now \mathbf{M} into an \mathcal{L}_s -structure \mathbf{M}_s on the same domain M . We encode $s^{\mathbf{M}}$ using the new dual symbols as follows: for $f \in k^n$ we let $\mathbf{M}_s \models R_s^f(\Delta_0, \dots, \Delta_k)$, if and only if there are $a_0, \dots, a_{n-1} \in M$ such that

$$\mathbf{M} \models s(a_0, \dots, a_{n-1}) \text{ and } a_i \in \Delta_{f(i)} \text{ for every } i \in n.$$

It can easily be checked that $\text{Aut}(\mathbf{M})$ can be fully recovered from $\text{Aut}(\mathbf{M}_s)$, that \mathbf{M} is projectively Fraïssé if and only if \mathbf{M}_s is, and that $\text{Aut}(\mathbf{M})$ and $\text{Aut}(\mathbf{M}_s)$ are equal as permutation groups on M .

There are cases of topological \mathcal{L} -structures which can be turned into purely direct structures. However, this is not the case always. The main observation is that if r is direct relation of arity k which belongs to \mathcal{L} and \mathbf{M} is a topological \mathcal{L} -structure then $r^{\mathbf{M}}$ is a set-wise invariant closed subset of M^k . Let now $K = 2^{\mathbb{N}}$ and let μ be the uniform probability measure on $2^{\mathbb{N}}$. The group $\text{Aut}(K, \mu)$ of all continuous measure preserving bijections can be easily seen to be a closed proper subgroup of $\text{Homeo}(K)$ which for every $n > 0$ leaves no proper subset of K^n invariant. Therefore, the canonical Fraïssé structure given by an application of Theorem 3.1 on $\text{Aut}(K, \mu)$ cannot be turned into a purely direct one.

5. COMPACT POLISH SPACES AS QUOTIENTS OF DUAL FRAÏSSÉ STRUCTURES

We fix a special binary relation symbol \mathbf{r} whose interpretation will always be a reflexive and symmetric closed relation. A formal relational language \mathcal{L} will be decorated with the subscript \mathbf{r} whenever $\mathbf{r} \in \mathcal{L}_{\mathbf{r}}$. Therefore, an $\mathcal{L}_{\mathbf{r}}$ -structure is always going to be a reflexive \mathbf{r} -graph perhaps with some extra structure. We say that an $\mathcal{L}_{\mathbf{r}}$ -structure \mathbf{K} is a pre-space if $\mathbf{r}^{\mathbf{K}}$ is moreover transitive and therefore an equivalence relation.

As we noted in the introduction T. Irwin and S. Solecki used in [ISS06] their notion of projective Fraïssé limit to express the pseudo-arc P as a quotient of a projective Fraïssé $\{\mathbf{r}\}$ -structure \mathbb{P} via $\mathbf{r}^{\mathbb{P}}$. Moreover, through their construction, the group $\text{Aut}(\mathbb{P})$ naturally embedded in $\text{Homeo}(P)$ as a dense subgroup. In [RC10],

R. Camerlo characterized all different projective Fraïssé classes¹ of $\{\tau\}$ -structures. Their limits are pre-spaces with quotients M/τ^M which vary between certain combinations of singletons, Cantor spaces and pseudo-arcs [RC10]. In [BK13], D. Bartošová and A. Kwiatkowska express the Lelek fan L as the quotient of a the projective Fraïssé limit \mathbb{L} of a certain class of directed graphs. Their limit \mathbb{L} can be seen again as pre-space in some \mathcal{L}_τ . Here again the group $\text{Aut}(\mathbb{L})$ naturally embedded in $\text{Homeo}(L)$ as a dense subgroup.

The purpose of this chapter is to show that under the notion of projective Fraïssé limit that we use here the same representation applies to every second-countable compact space Y . Since this is trivial for finite spaces, we will restrict ourselves to the case where Y is infinite.

The following proposition will be use in the proof of Theorem 5.2.

Proposition 5.1. *Let G, H be Polish groups and let S be a dense subgroup of G . Then any continuous homomorphism $f : S \rightarrow H$ extends to a continuous homomorphism $\tilde{f} : G \rightarrow H$.*

The proof of Proposition 5.1 is an easy exercise given that every Polish admits a compatible left-invariant metric d and given this metric we can define a new compatible complete metric D by $D(x, y) = d(x, y) + d(x^{-1}, y^{-1})$. For more details see page 6 of [BK96].

Theorem 5.2. *For every second-countable, compact space Y there is a projective Fraïssé pre-space \mathbf{K} such that:*

$$(K/\tau^K) \cong_{\text{homeo}} Y.$$

Moreover, the map $\text{Aut}(\mathbf{K}) \rightarrow \text{Homeo}(Y)$ induced by the quotient map $K \rightarrow (K/\tau^K)$ is a continuous group embedding with dense image in $\text{Homeo}(Y)$.

Proof. Let Y be an infinite compact Polish space and let H be a countable, dense subgroup of $\text{Homeo}(Y)$. In what follows, we define a countable Boolean algebra $(\mathcal{F}, 0_{\mathcal{F}}, 1_{\mathcal{F}}, \wedge, \vee, ')$ of closed subsets of Y as well as an action of H on \mathcal{F} via Boolean algebra automorphisms. Every set $F \in \mathcal{F}$ will be regular closed. Recall that an open set U is called regular open if $\text{int}(\overline{U}) = U$ and a closed set F is called regular closed if $\text{int}(\overline{F}) = F$. We define $0_{\mathcal{F}}, 1_{\mathcal{F}}$ and the operations $\wedge, \vee, '$ as follows:

- $0_{\mathcal{F}} = \emptyset$;
- $1_{\mathcal{F}} = Y$;
- $F_1 \wedge F_2 = \overline{\text{int}(F_1 \cap F_2)}$;
- $F_1 \vee F_2 = F_1 \cup F_2$;
- $F' = \overline{F^c}$.

The boolean algebra axioms are satisfied by the above configuration since \mathcal{F} consists of regular closed sets (see also [Hal63] for the boolean algebra of regular open sets).

¹He allows Fraïssé classes to lack hereditary property.

To construct the boolean algebra fix first a compatible complete metric d on Y . For every n chose a finite open cover $\{V_0^n, \dots, V_{k_n}^n\}$ of Y such $\text{diam}(V_i^n) < 1/n$ for every $i \in \{0, \dots, k_n\}$. Since \overline{V} is regular closed for every open V we have that the collection $\mathcal{J} = \{F_i^n : F_i^n = \overline{V_i^n}, n \in \mathbb{N}, 0 \leq i \leq k_n\}$ consists of regular closed sets. We define \mathcal{F} to be the least family of closed subsets of Y such that:

- (1) $\mathcal{J} \subset \mathcal{F}$;
- (2) \mathcal{F} is closed under the boolean operators $\wedge, \vee, '$ and
- (3) \mathcal{F} is closed under translation by elements of H , i.e., if $h \in H$ and $F \in \mathcal{F}$ then $h(F) \in \mathcal{F}$.

Notice that all these operations preserve regularity and since \mathcal{J} and H are countable \mathcal{F} is a countable family of regular closed sets. Notice that this implies that the only $F \in \mathcal{F}$ that has empty interior is the empty set. The group H is acting on \mathcal{F} with Boolean algebra automorphisms: for every $h \in H$ and $F \in \mathcal{F}$ let

$$h \cdot F = h(F).$$

Let $K = S(\mathcal{F})$ be the Stone space of all ultrafilters x on \mathcal{F} . This space comes with a topology whose basic clopen sets can be taken to be the sets of the form $\tilde{F} = \{x : F \in x\}$ for $F \in \mathcal{F}$. The space K is a compact, second-countable, and zero-dimensional. Let $p : K \rightarrow Y$ be the natural projection defined by:

$$\{p(x)\} = \bigcap_{F \in x} F.$$

The map p is continuous surjection with $p(\tilde{F}) = F$ for every $F \in \mathcal{F}$. We can turn now K to a $\{\mathfrak{r}\}$ -structure $K_{\mathfrak{r}}$ by setting $K_{\mathfrak{r}} \models \mathfrak{r}(x_0, x_1)$ if and only if $p(x_0) = p(x_1)$. It is immediate that $K_{\mathfrak{r}}$ is a pre-space and that $K/\mathfrak{r}^{K_{\mathfrak{r}}} = Y$.

Notice now that H is acting on K with homeomorphisms: for every $h \in H$ and $x \in K$

$$h \cdot x = \{h(F) : F \in x\} \in K.$$

This action is faithful since for every pair $y_0, y_1 \in Y$ there are $F_0, F_1 \in \mathcal{F}$ such that $y_0 \in \text{int}(F_0)$, $y_1 \in \text{int}(F_1)$ and $F_0 \cap F_1 = \emptyset$. Therefore, H embeds into $\text{Homeo}(K)$. We will denote this copy of H inside $\text{Homeo}(K)$ by H_K to distinguish it from H which is a subgroup of $\text{Homeo}(Y)$ and we will denote by T_0 the inverse of this embedding, i.e.,

$$T_0 : H_K \rightarrow H \quad \text{with} \quad \widetilde{T_0(h)(F)} = h(\tilde{F}), \quad \text{for every } F \in \mathcal{F}.$$

The map T_0 is also continuous. To see that, let $h \in H_K$ and let $V(L, U)$ be an open neighborhood of $T_0(h)$ in $\text{Homeo}(Y)$, i.e., $T_0(h)(L) \subset U$. Since the family $\{\text{int}(F) : F \in \mathcal{J}\}$ constitutes a basis of Y and since $T_0(h)(L)$ is compact, we can find $F_1, \dots, F_k \in \mathcal{J}$ such that $T_0(h)(L) \subseteq F_1 \cup \dots \cup F_k \subseteq U$. Let $F_0 = F_1 \vee \dots \vee F_k$, then both F_0 and $h^{-1}(F_0)$ belong to \mathcal{F} . Moreover, $V(\widetilde{h^{-1}(F_0)}, \tilde{F}_0)$ is an open neighborhood of h in H_K that is mapped via T_0 completely inside $V(L, U)$, proving that T_0 is continuous at h .

By applying the Theorem 3.1, we can endow K with a topological Fraïssé structure \mathbf{K}_0 in a purely dual language \mathcal{L} , such that $\overline{H_K} = \text{Aut}(\mathbf{K}_0)$ (the closure here is taken in $\text{Homeo}(K)$). By Proposition 5.1 the map T_0 extends to a continuous homomorphism $T : \text{Aut}(\mathbf{K}_0) \rightarrow \text{Homeo}(Y)$. We denote the image of $\text{Aut}(\mathbf{K}_0)$ under T by \hat{H} . Notice that \hat{H} lies densely in $\text{Homeo}(K)$ since $H < \hat{H} \leq \text{Homeo}(K)$, and since the same is true for H . Moreover, by the density of H_K in $\overline{H_K}$ the continuity of T and the fact that every $F \in \mathcal{F}$ has non-empty interior we get that for every $h \in \overline{H_K}$ and for every $F \in \mathcal{F}$ the following equality holds

$$(1) \quad \widetilde{T(h)}(F) = h(\tilde{F}).$$

We combine now the structures \mathbf{K}_0 and K_τ into one \mathcal{L}_τ -structure \mathbf{K} on domain K , where $\mathcal{L}_\tau = \mathcal{L} \cup \{\tau\}$. Notice that τ^K is invariant under $\text{Aut}(\mathbf{K}_0)$ since $(x_0, x_1) \in \tau^K$ if and only if for all $F_0, F_1 \in \mathcal{F}$ with $x_0 \in \tilde{F}_0$ and $x_1 \in \tilde{F}_1$ we have that $F_0 \cap F_1 \neq \emptyset$. Thus $\text{Aut}(\mathbf{K}) = \text{Aut}(\mathbf{K}_0) = \overline{H_K}$, every $\mathbf{A}_0 \in \text{Age}(\mathbf{K}_0)$ uniquely extends to an $\mathbf{A} \in \text{Age}(\mathbf{K})$ and \mathbf{K} is also a projective Fraïssé structure. The fact that $p(\tilde{F}) = F$ for every $F \in \mathcal{F}$ and the relation (1) above let us view $T : \text{Aut}(\mathbf{K}) \rightarrow \text{Homeo}(Y)$ as the homomorphism induced by the quotient $p : K \rightarrow Y$.

We are left to show that T is injective. Let $h \in \text{Aut}(\mathbf{M})$ so that $h \neq \text{id}_{\text{Aut}(\mathbf{M})}$. By the continuity of h we can find a non-empty F in \mathcal{F} so that $F \wedge h(F) = \emptyset$. Therefore, the interiors in Y of $p(F)$ and $p(h(F))$ do not intersect and because the interior in Y of every non-empty F in \mathcal{F} is non-empty we have that $T(h) \neq \text{id}_{\text{Homeo}(Y)}$. \square

We should remark here that the image \hat{H} of $\text{Aut}(\mathbf{K})$ under T is in general meager in $\text{Homeo}(Y)$. This can be seen as follows: first notice that as a corollary of Pettis theorem we have that if $f : G \rightarrow H$ is a Baire-measurable homomorphism between Polish groups and $f(G)$ is not meager, then f is open (see for example Theorem 1.2.6 [BK96]). Now notice that for $F \in \mathcal{F}$ the set $V(\tilde{F}, \tilde{F})$ is open in $\text{Homeo}(K)$ but the set $V(F, F)$ is rarely open in $\text{Homeo}(Y)$ (except if Y is zero-dimensional or it has very few homeomorphisms). Therefore T will fail in general to be an open map.

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